A note on one inverse spectral problem.

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Abstract

The note contains the proof of the uniqueness theorem for the inverse problem in the case of n-th order differential equation.

The inverse spectral problem is studied in papers of many authors ([1]–[8]). Extensive bibliographies for the inverse spectral problem can be found in [7]– [8].

Let's consider the spectral problem for the common differential differential equation:

$$F(x, y(x), y'(x), \dots, y^{(n)}, q_1(x), q_2(x), \dots, q_m(x), \lambda) = 0$$
 (1)

with common boundary conditions

$$U_j(y(x), \lambda, a_0, a_1, \dots, a_s) = 0, \qquad j = 1, \dots, n.$$
 (2)

Here $x \in [0,1]$, λ is eigenvalue parameter, q_i $(i=1,\ldots m)$ are uncertain factors of the equation, a_i $(i=0,\ldots,s)$ are uncertain constants of boundary conditions, $q_i \in C^1[0,1]$ $(i=1,\ldots m)$, $a_i \in \mathbb{C}$ $(i=0,\ldots,s)$.

The spectral problem defined by equalities (1)–(2) we shall name F. Along with the problem F we shall consider m problems: A_i :

$$-y'' + q_i(x) y = \lambda y,$$

$$y'(0) = 0,$$

$$y'(1) = 0.$$

and one more problem A_{m+1} :

$$y'' + 3y' + 2\lambda^2 y = 0, (3)$$

$$y(0) = 0, (4)$$

$$y'(1) + a(\lambda) \cdot y(1) = 0, \tag{5}$$

Theorem.

If eigenvalues of the problems A_i and \widetilde{A}_i (i=1, 2, ..., m+1) coincide with their algebraic multiplicities, then the factors of the equations and the constant in the boundary conditions of the problems F and \widetilde{F} coincide, that is $q_i(x) \equiv \widetilde{q}_i(x)$, $a_k = \widetilde{a}_k$ i = 1, 2, ..., m k = 1, 2, ..., s.

Proof. It follows from Ambarzumijan's theorem ([1]) that the equality $q_i(x) = \widetilde{q}_i(x)$ is true.

Functions $y_1(x, \lambda) = -e^{2\lambda x} + 2e^{\lambda x}$, $y_2(x, \lambda) = \frac{1}{\lambda} (e^{2\lambda x} - e^{\lambda x})$ are solutions of the differential equation (3), satisfying

$$y_1(0, \lambda) = 1, \quad y_1'(0, \lambda) = 0, \qquad y_2(0, \lambda) = 0, \quad y_2'(0, \lambda) = 1.$$
 (6)

Let $A(\lambda)$ be a polynomial $a_0 + a_1 \lambda + a_2 \lambda^2 + \ldots + a_s \lambda^s$.

The eigenvalues λ_i of the problems (3)–(5) are the roots of a characteristic determinant, therefore its satisfy to the following equation:

$$\Delta(\lambda) = \frac{1}{\lambda} \left(e^{2\lambda} - e^{\lambda} \right) + a(\lambda) \left(-e^{2\lambda} + 2e^{\lambda} \right) = 0.$$
 (7)

This function has infinite number of the radicals. (For this reason the equation was selected by such: $y'' + 3y' + 2\lambda^2 y = 0$. Generally speaking it was possible to select any equation having not less s pairwise different nonzero eigenvalues.)

From (6) we have

$$1 \cdot a_0 + \lambda_i \cdot a_1 + \lambda_i^2 \cdot a_2 + \ldots + \lambda_i^s \cdot a_s = -\frac{e^{2\lambda_i} - e^{\lambda_i}}{-\lambda_i e^{2\lambda_i} + 2\lambda_i e^{\lambda_i}}.$$
 (8)

The equalities (8) is a system of (s+1) linear equaions having (s+1) unknown $a_0, a_1, a_2, \ldots, a_s$ of the boundary condition. The determinant of this system is the Vandermonde determinant. As the eigenvalues λ_i are pairwise different and are not equal to zero, the Vandermonde determinant is not equal to zero. Therefore system (8) has a unique solution. From here Follows, that the constants $a_0, a_1, a_2, \ldots, a_s$ are determined univalently.

The theorem is proved.

References

[1] Ambarzumijan V. A. Uber eine Frage der Eigenwerttheorie // Zeitshrift fur Physik. — 1929. — No. 53. — S. 690–695

- [2] Borg G. Eine Umkehrunrung der Sturm-Liouvillschen Eigenwertanfgabe. Bestimmung der Differentialgleichung durch die Eigenwarte // Acta Mathematica. 1946. 78, N 2. S. 1–96.
- [3] Levinson N. The inverse Sturm–Liouville problem // Math. Tidsskr. Ser.B. 1949. V. 13. P. 25–30.
- [4] Leybazon Z. L. An inverse problem of the spectral analysis of order differential operators of higher order [in Russian] Trudy. Moskov. Mat. Obsc. 1966. V. 15. P. 70–144 = Trans. Moscow Math. Soc. 1966, 78– 163. MR 34 # 4951.
- [5] Sadovnichiy V.A. Uniqueness of the solution to the inverse problem in the case of a second-order equation with the indecomposable boundary conditions, regularized sums of some of the eigenvalues. Factorization of the characteristic determinant. [in Russian] Dokl. Akad. Nauk SSSR. 1972. Tom 206. No.2. P. 293–296. = Soviet Math. Dokl. 1972. Vol. 13. No. 5. P. 1220–1223.
- [6] Hochstadt H. The inverse Sturm–Lioville problem // Comm. Pure Appl. Math. 1973. V. 26. P. 715–729
- [7] Naymark M. A. Linear differential operators. II. Ungar. New York. 1968.
- [8] Levitan B.M. The inverse Sturm–Liouville problems and applications [in Russian] Nauka, Moscow. 1984.